

THE PERIOD-INDEX PROBLEM OF THE CANONICAL GERBE OF SYMPLECTIC AND ORTHOGONAL BUNDLES

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ABSTRACT. We consider regularly stable parabolic symplectic and orthogonal bundles over an irreducible smooth projective curve over an algebraically closed field of characteristic zero. The morphism from the moduli stack of such bundles to its coarse moduli space is a μ_2 -gerbe. We study the period and index of this gerbe, and solve the corresponding period-index problem.

1. INTRODUCTION

Let X be a smooth, projective, irreducible curve over an algebraically closed field k of characteristic zero. A symplectic (respectively, orthogonal) vector bundle with parabolic structure on X consists of a parabolic vector bundle (E_*, B) with a skew-symmetric (respectively, symmetric) pairing

$$E_* \otimes E_* \longrightarrow \mathcal{O}_X,$$

satisfying a nondegeneracy condition (see Section 2). Note that the above tensor product above is a tensor product in the category of parabolic vector bundles.

A regularly stable symplectic (respectively, orthogonal) parabolic bundle is one whose automorphism group coincides with the center of the symplectic (respectively, orthogonal) group. Let $\mathrm{Bun}_G^{\mathrm{rs}}$ be the moduli stack of regularly stable symplectic or orthogonal parabolic bundles, and let M_G^{rs} be the corresponding coarse moduli space. We have a μ_2 -gerbe

$$\mathrm{Bun}_G^{\mathrm{rs}} \longrightarrow M_G^{\mathrm{rs}}.$$

The purpose of this paper is to study the period-index problem for this gerbe. The center of $\mathrm{SO}(2n+1)$ is trivial, hence in this case $\mathrm{Bun}_G^{\mathrm{rs}} = M_G^{\mathrm{rs}}$. Therefore, we will assume that the rank in the orthogonal case is even.

The index is computed in Theorem 7.4. The main idea is to degenerate the gerbe to a highly singular point (see Proposition 2.4) and to use Luna's étale slice theorem to study the geometry of this stack over the moduli space here.

In Section 2, we briefly recall the definition and properties of parabolic bundles on X , and recall the construction of their moduli spaces. In Proposition 2.2 we compute the period of the canonical gerbe in most cases. We also discuss an application of Luna's étale slice theorem to our case using Proposition 2.4. In Section 3, after a brief overview of twisted sheaves, we give an upper bound for the index of the canonical gerbe.

In Sections 4 and 5, we discuss the concept of a stable central simple algebra with involution, and prove that for a stable central simple algebra with involution over a field F , there exists a morphism $\mathrm{Spec} F \longrightarrow Z^s/G^{\mathrm{ad}}$, where G denotes an appropriate symplectic or orthogonal group. Finally, in Section 6, we prove the existence of lower bounds for the index of the canonical gerbe. This gives the index completely in the symplectic case, and it gives a very strong lower bound for

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the orthogonal case. The appendices discuss a technical result used in the proof, and symplectic or orthogonal involutions on a central simple algebra.

1.1. Conventions.

- We work over an algebraically closed base field k of characteristic zero.
- By G we denote $\mathrm{Sp}(2n)$ or $\mathrm{SO}(2n)$, and G^{ad} denotes the adjoint form of G , i.e. $\mathrm{PSp}(2n)$ or $\mathrm{PSO}(2n)$. In the case of $G = \mathrm{SO}(2n)$, we assume that $n \geq 2$. Also, $G(j)$ denotes $\mathrm{Sp}(j)$ or $\mathrm{SO}(j)$. By \mathfrak{g} we denote the Lie algebra of G , while $\mathfrak{g}(j)$ denotes the Lie algebra of $G(j)$.
- X denotes an irreducible smooth projective curve of genus $g(X) \geq 2$.
- $D := \{x_1, \dots, x_n\} \subset X$ is the ordered set of parabolic points on X .
- M_G^{rs} , M_G^s and M_G^{ss} denote the coarse moduli spaces of regularly stable, stable and semistable parabolic G -bundles, while Bun_G^{rs} , Bun_G^s and Bun_G^{ss} are similarly defined stacks of parabolic G -bundles.
- \mathcal{E} is the universal parabolic bundle on $X \times \mathrm{Bun}_G^*$ where “ $*$ ” stands for rs , s or ss .
- $p : X \times \mathrm{Bun}_G^{rs} \rightarrow \mathrm{Bun}_G^{rs}$ denotes the natural projection.

2. THE MODULI SPACE OF PARABOLIC BUNDLES

2.1. Parabolic G -Bundles over a Curve. Let E_* be a parabolic vector bundle on X with parabolic structure over D . A *bilinear form* on E_* is a homomorphism of parabolic bundles $B : E_* \otimes E_* \rightarrow \mathcal{O}_X$, where \mathcal{O}_X is the trivial line bundle with trivial parabolic structure (this means that there are no nonzero parabolic weights). The parabolic vector bundle $E_* \otimes E_*^\vee$, where E_*^\vee is the parabolic dual, is given by the sheaf of endomorphisms of E_* compatible with the parabolic structure. We have a natural homomorphism of parabolic bundles $h : \mathcal{O}_X \rightarrow E_* \otimes E_*^\vee$ (as before, \mathcal{O}_X is the trivial line bundle with trivial parabolic structure) that sends any locally defined function f to the locally defined endomorphism $s \mapsto f \cdot s$ of E_* . Given a bilinear form B on E_* , the composition

$$(2.1) \quad E_* = E_* \otimes \mathcal{O}_X \xrightarrow{\mathrm{Id} \otimes h} E_* \otimes E_* \otimes E_*^\vee \xrightarrow{B \otimes \mathrm{Id}} E_*^\vee$$

will be denoted by \widehat{B} .

A *symplectic* parabolic vector bundle is a pair (E_*, B) , where E_* is a parabolic vector bundle, and B is a skew-symmetric bilinear form on E_* such that the above homomorphism \widehat{B} is an isomorphism. A symplectic parabolic vector bundle (E_*, B) of rank $2n$ will be called a *parabolic* $\mathrm{Sp}(2n)$ -bundle. An *orthogonal* parabolic vector bundle (E_*, B) is defined similarly, with a symmetric bilinear form B on E_* . An orthogonal parabolic vector bundle (E_*, B) of rank n will be called a *parabolic* $\mathrm{SO}(n)$ -bundle.

If \widehat{B} is an isomorphism, then $\mathrm{par-deg}(E_*) = \mathrm{par-deg}(E_*^\vee) = -\mathrm{par-deg}(E_*)$. Hence symplectic and orthogonal parabolic bundles have parabolic degree zero.

An *isomorphism* between two parabolic G -bundles (E_*, B) and (E'_*, B') is an isomorphism of parabolic vector bundles $\phi : E_* \rightarrow E'_*$ such that the following diagram commutes:

$$\begin{array}{ccc} E_* \otimes E_* & \xrightarrow{\phi \otimes \phi} & E'_* \otimes E'_* \\ \downarrow B & & \downarrow B' \\ \mathcal{O}_X & \xrightarrow{\mathrm{Id}} & \mathcal{O}_X \end{array}$$

A parabolic G -bundle (E_*, B) is called *stable* (respectively, *semistable*) if for any nonzero proper parabolic subbundle $F_* \subset E_*$ with $B(F_* \otimes F_*) = 0$, the inequality

$$\frac{\text{par-deg}(F_*)}{\text{rank}(F_*)} < \frac{\text{par-deg}(E_*)}{\text{rank}(E_*)} \quad (\text{respectively, } \frac{\text{par-deg}(F_*)}{\text{rank}(F_*)} \leq \frac{\text{par-deg}(E_*)}{\text{rank}(E_*)})$$

holds.

A stable parabolic G -bundle (E_*, B) is called *regularly stable* if it has no automorphism other than $\pm \text{Id}_{E_*}$.

Remark 2.1. Let (E_*, B) be a parabolic $\text{SO}(a)$ -bundle. Then for each positive integer b , the parabolic vector bundle $(E_*, B)^{\oplus b}$ has a natural structure of a parabolic $\text{SO}(ab)$ -bundle; and if b is even, then $(E_*, B)^{\oplus b}$ also has the structure of an $\text{Sp}(ab)$ -bundle. Indeed, $(E_*, B)^{\oplus b} = (E_*, B) \otimes k^{\oplus b}$. Now put the standard symplectic or orthogonal structure on $k^{\oplus b}$. The orthogonal structure B on E_* and the symplectic or orthogonal structure on $k^{\oplus b}$ together define a symplectic or orthogonal structure on the tensor product $(E_*, B) \otimes k^{\oplus b}$.

2.2. A Construction of the Moduli Space. Fix a parabolic type, meaning parabolic weights and quasi-parabolic filtration types. We assume that the parabolic type is so chosen that there is a parabolic G -bundle of the given type. This requires the parabolic type to be compatible with the G -structure of the bundles. Let M_G^{ss} be the moduli space of semistable parabolic G -bundles of the given parabolic type.

We now briefly describe the construction of this moduli space M_G^{ss} as done in [2]. Given a parabolic type, there is an associated Galois cover $Y \rightarrow X$ with finite Galois group Γ . Now [2, Theorem 4.3] gives an equivalence between parabolic G -bundles on X and (Γ, G) -bundles on Y . Using this equivalence, the moduli space of semistable parabolic G -bundles on X is constructed in the following manner. Choose an integer m_0 such that for $m \geq m_0$ and any semistable Γ -bundle \mathcal{E} on Y of rank $2n$ and trivial determinant one has $h^i(\mathcal{E}(m)) = 0$ for $i > 0$ and $\mathcal{E}(m)$ is globally generated. Let $N = h^0(\mathcal{E}(m))$. Let $P(t) = 2n(t+1) - g(X)$ be the Hilbert polynomial of \mathcal{E} . Then there is a well-known Quot scheme Q , constructed by Grothendieck, which parametrizes quotients of $\mathcal{O}_Y(-m)^N$ with Hilbert polynomial P . The finite group Γ acts on Q . Let \mathcal{G} denote the group of Γ -invariant automorphisms of $\mathcal{O}_Y(-m)^N$. This is a reductive group by [14]. Let Q^Γ denote the Γ -invariant locus in Q . There exists a nonempty open subset \mathcal{R}^{ss} of Q^Γ consisting of semistable bundles. One then constructs a scheme Q_G with a \mathcal{G} -action together with a \mathcal{G} -equivariant morphism $Q_G \rightarrow \mathcal{R}^{\text{ss}}$. Then the GIT quotient of Q_G by \mathcal{G} is the moduli space M_G^{ss} . We note that there exist open subsets $\mathcal{R}^{\text{rs}} \subset \mathcal{R}^s \subset \mathcal{R}^{\text{ss}}$ consisting of regularly stable and stable parabolic G -bundles; and their GIT quotients under \mathcal{G} , $M_G^{\text{rs}} \subset M_G^s \subset M_G^{\text{ss}}$ are the corresponding coarse moduli spaces. (We refer the reader to [2, Section 5] for the details of this construction.)

2.3. The period of the canonical gerbe. Let d be the degree of the vector bundle underlying a parabolic G -bundle in M_G^{rs} . Then the total parabolic weight is $-d$ since the parabolic degree is zero. At $x_i \in D$, let $\{n_{1,i}, \dots, n_{\ell_i,i}\}$ be the multiplicities of the parabolic weights at x_i . (Recall that these are the dimensions of the graded pieces of the quasi-parabolic filtration at x_i .)

Define

$$(2.2) \quad \epsilon := \text{g.c.d.}\{d, 2n, \{n_{1,i}, \dots, n_{\ell_i,i}\}_{i=1}^n\}.$$

Proposition 2.2. *If ϵ is odd, there is a Poincaré vector bundle on $X \times M_G^{\text{rs}}$. If ϵ is even, there is no Poincaré vector bundle on $X \times M_{\text{Sp}(2n)}^{\text{rs}}$. If $\epsilon \geq 4$ is even, there is no Poincaré vector bundle on $X \times M_{\text{SO}(2n)}^{\text{rs}}$. If $\epsilon = 2$, there is no Poincaré vector bundle on $X \times M_{\text{SO}(4n)}^{\text{rs}}$.*

Proof. Recall that Bun_G^{rs} is the moduli stack of regularly stable parabolic G -bundles, and $\mathcal{E} \rightarrow X \times \text{Bun}_G^{\text{rs}}$ is the universal parabolic bundle. Let

$$(2.3) \quad f : \text{Bun}_G^{\text{rs}} \rightarrow M_G^{\text{rs}}$$

be the morphism to the coarse moduli space.

For any integer m , its image in $\mathbb{Z}/2\mathbb{Z}$ will be denoted by \overline{m} . Note that for each integer $n_{\ell,i}$ in (2.2), there is a line bundle of weight $\overline{n_{\ell,i}}$ on Bun_G^{rs} . Similarly, there is a line bundle of weight $\overline{d + 2n(1 - g(X))} = \overline{d}$ on Bun_G^{rs} given by the determinant of cohomology of \mathcal{E} .

First assume that ϵ is odd. Then there is a line bundle L of weight $\overline{1}$ on Bun_G^{rs} . Hence the vector bundle $\mathcal{E} \otimes p^*(L)$ on $X \times \text{Bun}_G^{\text{rs}}$ has weight zero; therefore, it descends to $X \times M_G^{\text{rs}}$ as a Poincaré vector bundle.

Now assume that ϵ is even; if $G = \text{SO}(2n)$, then assume that $\epsilon \geq 4$. We will show that there is no Poincaré vector bundle on $X \times M_G^{\text{rs}}$.

Let us first consider the $\text{Sp}(2n)$ case. If (V_*, B_0) is a parabolic $\text{SO}(n)$ -bundle, then the parabolic direct sum $V_* \oplus V_* = V_* \otimes k^{\oplus 2}$ has a symplectic structure. Indeed, the symmetric bilinear form B_0 on V_* and the standard symplectic form on $k^{\oplus 2}$ together define a parabolic $\text{Sp}(2n)$ -structure \tilde{B}_0 on $V_* \otimes k^{\oplus 2}$. Since ϵ is a multiple of 2, there is stable parabolic $\text{SO}(n)$ -bundle (V_*, B_0) such that the corresponding parabolic $\text{Sp}(2n)$ -bundle $(V_* \otimes k^{\oplus 2}, \tilde{B}_0)$ is a parabolic $\text{Sp}(2n)$ -bundle of the given type; this parabolic $\text{Sp}(2n)$ -bundle is semistable because (V_*, B_0) is semistable.

Now consider the $\text{SO}(2n)$ case. We can then go through the same construction as follows. We choose a stable parabolic $\text{SO}(2n/\epsilon)$ -bundle (V_*, B_0) such that the corresponding parabolic $\text{SO}(2n)$ -bundle $(V_* \otimes k^{\oplus \epsilon}, \tilde{B}_0)$ is a semistable parabolic $\text{SO}(2n)$ -bundle of the given type; here $k^{\oplus \epsilon}$ is equipped with the standard orthogonal form.

Finally, consider $\text{SO}(4n)$ with $\epsilon = 2$. In this case there is a stable parabolic $\text{Sp}(2n)$ -bundle (V_*, B_0) such that $(V_* \otimes k^{\oplus 2}, \tilde{B}_0)$ is a semistable parabolic $\text{SO}(4n)$ -bundle, where $k^{\oplus 2}$ is equipped with the standard symplectic form.

The automorphism group of $(V_* \otimes k^{\oplus 2}, \tilde{B}_0)$ in the $\text{Sp}(2n)$ case, or $(V_* \otimes k^{\oplus \epsilon}, \tilde{B}_0)$ and $(V_* \otimes k^{\oplus 2}, \tilde{B}_0)$ in the two $\text{SO}(2n)$ cases, contains $\text{Sp}(2)$ or $\text{SO}(\epsilon)$, and hence the center $\pm \text{Id}$ of this $\text{Sp}(2)$ or $\text{SO}(\epsilon)$ is contained in the automorphism group. Since $\text{Sp}(2)$ or $\text{SO}(\epsilon)$ does not have any nontrivial character, by [3, p. 1286, Theorem 2.2] we conclude that there is no Poincaré vector bundle on $X \times M_G^{\text{rs}}$. \square

Remark 2.3. If ϵ is odd, Proposition 2.2 shows that the canonical μ_2 -gerbe $\text{Bun}_G^{\text{rs}} \rightarrow M_G^{\text{rs}}$ is neutral. Hence its period and index are 1. Unless otherwise stated, from now on we assume that ϵ is even and that $\epsilon \geq 4$ in the $\text{SO}(2n)$ case (note that this condition is automatically satisfied if there are no parabolic points).

Proposition 2.4. *Let $r = 2n/\epsilon$. There is a regularly stable parabolic $\text{SO}(r)$ -bundle (E_*, B) such that $(E_*, B)^{\oplus \epsilon} \in M_{\text{SO}(2n)}^{\text{ss}}$. Similarly, there is a regularly stable parabolic $\text{SO}(r)$ -bundle (E_*, B) such that $(E_*, B)^{\oplus \epsilon} \in M_{\text{Sp}(2n)}^{\text{ss}}$.*

Proof. Fix rank, degree, quasi-parabolic types and parabolic weights. The necessary and sufficient condition for the existence of a parabolic orthogonal bundle with this data is the following:

- (1) The parabolic degree is zero,
- (2) At each parabolic point, if π is a parabolic weight, then $1 - \pi$ is also a parabolic weight, and

- (3) At each parabolic point, the multiplicity of any parabolic weight π coincides with the multiplicity of the parabolic weight $1 - \pi$.

The necessary and sufficient condition for the existence of a parabolic symplectic bundle with this data is the following:

- (1) The rank is even,
- (2) The parabolic degree is zero,
- (3) At each parabolic point, if π is a parabolic weight, then $1 - \pi$ is also a parabolic weight, and
- (4) At each parabolic point, the multiplicity of any parabolic weight π coincides with the multiplicity of the parabolic weight $1 - \pi$.

If there is a parabolic orthogonal (respectively, symplectic) bundle, then there is a semistable parabolic orthogonal (respectively, symplectic) bundle. This follows from the fact that the stratum of parabolic orthogonal or symplectic bundles with given Harder-Narasimhan filtration type has dimension less than the dimension of the moduli space. Again for dimension reasons, there is a regularly stable bundle (E_*, B) as in the statement of the proposition. \square

2.4. Luna's Étale Slice Theorem. We follow the exposition in [8].

Definition 2.5. Let H be a reductive linear algebraic group. An H -equivariant morphism $S \rightarrow T$ of varieties is said to be *strongly étale* if $S//H \rightarrow T//H$ is étale.

Theorem 2.6. Suppose that H acts on a smooth quasi-projective variety S and the action is linearized with respect to some very ample line bundle. Let $s \in S$ be a closed point with stabilizer H_s and closed orbit. Then there is an H -stable open subset $U \subseteq S$, containing s , and $V \subseteq U$ a H -stable smooth subvariety, such that if N_s is the normal space to the orbit $H \cdot s$ at s , then we have an equivariant diagram of strongly étale morphisms

$$\begin{array}{ccc} & H \times_{H_s} V & \\ \swarrow & & \searrow \\ U & & W \subseteq H \times_{H_s} N_s \end{array}$$

Proof. See [8, page 27]. \square

Example 2.7. We apply Theorem 2.6 to Q_G . For s , we take a point corresponding to $(E_*, B)^{\oplus \epsilon} = (E_*, B) \otimes k^{\oplus \epsilon}$ as in Proposition 2.4. Recalling notation from Section 1.1, we have $H_s = G(\epsilon)$. Take $V \subseteq Q_G$ as in the theorem.

Recall the discussion at the start of Section 2.2. There is a diagram

$$(2.4) \quad \begin{array}{ccccc} U & \longleftarrow & \mathcal{G} \times_{H_s} V & \longrightarrow & W \subseteq \mathcal{G} \times_{H_s} N_s \\ \downarrow & & \downarrow & & \downarrow \\ [U/\mathcal{G}] & \longleftarrow & [V/H_s] & \longrightarrow & [W/\mathcal{G}] \subseteq [N_s/H_s] \\ \downarrow & & \downarrow & & \downarrow \\ U//\mathcal{G} & \xleftarrow{\pi_W} & V//H_s & \xrightarrow{\pi_U} & W//\mathcal{G} \subseteq N_s//H_s \end{array}$$

Generically, $U \rightarrow U//\mathcal{G}$ is a \mathcal{G}/μ_2 -bundle. As μ_2 acts trivially on V , a similar statement is true for the middle composite. Generically $W \rightarrow W//\mathcal{G}$ is also a \mathcal{G}/μ_2 -bundle; see Proposition 2.9 below. It follows that the middle row consists, generically, of μ_2 gerbes over the bottom row.

Proposition 2.8. \mathcal{R}^{ss} is smooth.

Proof. See Remark 5.6 of [2]. □

For a parabolic G -bundle (E_*, B) , let $\mathcal{E}nd(E_*)$ be the subsheaf of $\mathcal{E}nd(E)$ defined by the sheaf endomorphisms preserving the quasi-parabolic filtrations. (So $\mathcal{E}nd(E_*)$ is the vector bundle underlying the parabolic tensor product $E_* \otimes E_*^\vee$.) Let

$$\mathcal{E}nd_B(E_*) \subset \mathcal{E}nd(E_*)$$

be the subbundle defined by the sheaf of endomorphisms β such that $B(\beta(v) \otimes w) + B(v \otimes (w)) = 0$ for all locally defined sections v and w of E .

Proposition 2.9. *Consider $s \in \mathcal{R}^{ss}$ as in Example 2.7.*

- (1) *The stabilizer of s inside \mathcal{G} is $G(\epsilon)$.*
- (2) *The normal space to $\text{orb}(s)$ at s can be identified with*

$$H^1(X, \mathcal{E}nd_B(E_*)) \otimes \mathfrak{g}(\epsilon).$$

- (3) *The natural action of the stabilizer on the normal bundle can be identified with the adjoint action on the $\mathfrak{g}(\epsilon)$ factor above.*

Proof. Recall that $(E_*, B)^{\oplus \epsilon}$ in Example 2.7 is regularly stable. In view of the definition of a regularly stable bundle, the proposition follows. □

Remark 2.10. Denote $g = h^1(X, \mathcal{E}nd_B(E_*))$. Note that $g \geq 2$. We write

$$Z = Z(\mathfrak{g}(\epsilon), g) = \mathfrak{g}(\epsilon)^{\otimes g}$$

for the normal space to $\text{orb}(s)$ at s as in Proposition 2.9. This is the normal space that occurs in the diagram (2.4).

Set $\Lambda = k\langle z_1, z_2, \dots, z_g \rangle$ be a polynomial ring in g non-commuting variables. A closed point of Z determines a Λ -module structure on k^ϵ . Denote by Q the standard symplectic or orthogonal form on k^{2n} . There is an open subset Z^s of Z consisting of points where the corresponding Λ -module has no nontrivial isotropic submodules. This coincides with stable locus for the adjoint action of $G(\epsilon)$, see [6, Proposition 4.2].

Using the notation of (2.4) we define the following loci :

$$\begin{aligned} (V//H_s)^t &= \pi_U^{-1}((U//\mathcal{G})^{\text{rs}}) \cap \pi_W^{-1}(Z^s/H_s) \\ (Z^s/H_s)^t &= \pi_W((V//H_s)^t) \\ (U//\mathcal{G})^t &= \pi_U((V//H_s)^t) \end{aligned}$$

As the two varieties on the corners of the bottom row in (2.4) are irreducible, and all maps are étale, these are non-empty open sets.

We have a marked point 0 of the quotient $U//\mathcal{G}$, corresponding to the bundle in Proposition 2.4. The diagram (2.4) induces an isomorphism

$$\widehat{\mathcal{O}_{U//\mathcal{G}, 0}} \cong \widehat{\mathcal{O}_{Z//G(\epsilon), 0}}.$$

We will denote this ring by

$$\widehat{\mathcal{O}}_0.$$

Finally we construct an open subscheme $\text{Spec}(\widehat{\mathcal{O}}_0)^t$ of $\text{Spec}(\widehat{\mathcal{O}}_0)$ by the following Cartesian diagram :

$$\begin{array}{ccc} \text{Spec}(\widehat{\mathcal{O}}_0)^t & \longrightarrow & (V//H_s)^t \\ \downarrow & & \downarrow \\ \text{Spec}(\widehat{\mathcal{O}}_0) & \longrightarrow & (V//H_s) \end{array}$$

Proposition 2.11. *The classes of the three μ_2 -gerbes defined in (2.4) are the same inside*

$$\mathrm{Br}(\mathrm{Spec}(\widehat{\mathcal{O}_0})^t)$$

Proof. This follows from [9, Chapter IV, 2.3.18]. \square

3. TWISTED SHEAVES

Consider a gerbe $\mathfrak{G} \rightarrow S$ banded by a sheaf of abelian groups A that is a subgroup of \mathbb{G}_m . A coherent sheaf \mathcal{F} on \mathfrak{G} has two actions of A on it, the inertial action and a second action by viewing A as a subsheaf of $\mathcal{O}_{\mathfrak{G}}$. A *twisted sheaf* on \mathfrak{G} is a coherent sheaf where these two actions coincide.

If \mathcal{F} is a locally free twisted sheaf on \mathfrak{G} then $\mathcal{F} \otimes \mathcal{F}^\vee$ descends to a Azumaya algebra on S whose Brauer class is the same as the Brauer class of \mathfrak{G} , see the proof of [11, Proposition 3.1.2.1].

Example 3.1. The natural action of $G(\epsilon)$ on k^ϵ produces a twisted sheaf on $[Z/G(\epsilon)]$. Explicitly, the action of $G(\epsilon)$ on Z extends to an action on the trivial bundle $Z \times k^\epsilon$. The Azumaya algebra on $Z^s // G(\epsilon)$ associated to this twisted sheaf will be denoted \mathcal{B} . The corresponding trivial Azumaya algebra on Z^s will be denoted \mathcal{A} . Note that \mathcal{B} pulls back to \mathcal{A} under π .

Twisted sheaves are a useful tool for understanding the difference between the period and the index. Let us assemble the pertinent results.

Proposition 3.2. *When $S = \mathrm{Spec}(K)$ in the above situation the period divides the index and the period and index have the same prime factors.*

Proof. This is well known; for example, see [7]. \square

Proposition 3.3. *Let $\mathfrak{G} \rightarrow \mathrm{Spec}(K)$ be a \mathbb{G}_m -gerbe over a field. Then the index of \mathfrak{G} divides m if and only if there is a locally free twisted sheaf on \mathfrak{G} of rank m .*

Proof. See [11, Proposition 3.1.2.1]. \square

Now consider the canonical μ_2 -gerbe $\mathrm{Bun}_G^{\mathrm{rs}} \rightarrow M_G^{\mathrm{rs}}$. In Remark 2.3, we have assumed that ϵ is even. Hence the period of the canonical gerbe is 2, and by Proposition 3.2, its index (over the function field of M_G^{rs}) is a power of 2.

From now on, we will write $\epsilon = 2m$.

Proposition 3.4. *The index of the canonical gerbe $\mathrm{Bun}_G^{\mathrm{rs}} \rightarrow M_G^{\mathrm{rs}}$ divides ϵ .*

Proof. To see that the index of the canonical gerbe divides the $n_{k,i}$ corresponding to a point x_i , one considers the restriction of the universal parabolic bundle on $X \times M_G^{\mathrm{rs}}$ to $x_i \times M_G^{\mathrm{rs}}$ and takes the graded piece corresponding to $n_{k,i}$.

To complete the proof, we have to produce a twisted sheaf of rank d on $\mathrm{Bun}_G^{\mathrm{rs}}$. For $k \gg 0$, one has $R^1 p_*(\mathcal{E} \otimes p^* \mathcal{O}_X(k)) = 0$. Since we are over the regularly stable locus, $p_*(\mathcal{E})$ is a twisted sheaf of rank $d + 2n(1 - g)$. This finishes the proof by Proposition 3.3. \square

Corollary 3.5. *Assume $G = \mathrm{SO}(4n)$ and $\epsilon = 2$. Then the period and index of the canonical gerbe are both 2.*

Proof. By Proposition 2.2, there is no Poincaré vector bundle on $X \times M_{\mathrm{SO}(4n)}^{\mathrm{rs}}$. Hence, the period of the canonical gerbe is 2. By Proposition 3.4, the index divides $\epsilon = 2$. The result follows. \square

4. STABLE CENTRAL SIMPLE ALGEBRAS WITH INVOLUTION

Definition 4.1. Let A be a central simple algebra of degree $(2m)^2$ with involution σ over a field F . An F -subalgebra $B \subseteq A$ is called *parabolic* if for some finite field extension K/F that splits (A, σ) , and an isomorphism $\phi : A_K \longrightarrow \text{End}(K^{2m}, Q_0)$, where Q_0 is the standard symplectic or orthogonal form, B_K leaves a (nontrivial) totally isotropic subspace $W \subseteq K^{2m}$ invariant.

Proposition 4.2. *The definition above is independent of the extension K/F and $\phi : A_K \longrightarrow \text{End}(K^{2m}, Q_0)$.*

Proof. One easily reduces the question to the following situation: we have a finite field extension L/K and a splitting

$$\alpha_K : A_K \longrightarrow \text{End}(K^{2m}, Q_0)$$

such that upon base extension to L there is an isotropic subspace W of L^{2m} preserved by B_L . The result now follows from Lemma 4.3. \square

Lemma 4.3. *Let F_1/F_2 be an extension of fields, and let $P \subseteq \text{GL}(m, F_2)$ be a subgroup that becomes parabolic under base extension to F_1 , preserving a subspace $V \subseteq F_1^m$ of dimension m' . Then there exists a subspace $V' \subseteq F_2^m$ of dimension m' , base extending to V , that is left invariant by P .*

Proof. Since P_{F_1} preserves V , the action of P_{F_1} on $\text{Grass}(m', m)$ has a fixed point, which we will denote by Q_1 . Since this action is obtained from the action of P on $\text{Grass}(m', m)$, it follows that the action must also have a fixed point Q_2 which gives Q_1 upon base extension. Hence there must be a P -invariant subspace $V' \subseteq F_2^m$ base extending to V . \square

We now give the definition of a stable central simple algebra (CSA for short) with involution. We remind the reader that the number $g \geq 2$ was defined in Remark 2.10.

Definition 4.4. Let A be a central simple algebra of degree $(2m)^2$ with involution σ over a field F . Let $x_1, \dots, x_g \in A$ be elements such that $\sigma(x_i) = -x_i$. The triple (A, σ, x_i) is called a *stable central simple algebra with involution* over F if the F -subalgebra of A generated by x_i is not contained in a parabolic subalgebra.

Remark 4.5. Note that the degree $2m$ of the central simple algebra is implicit in the definition.

Example 4.6. Recall the construction of the Azumaya algebra \mathcal{B} from Example 3.1. We now describe its construction in more detail to explain how the “universal stable CSA with involution” is formed.

Recall that there is an action of $G(\epsilon)$ on Z by conjugation. Consider the split Azumaya algebra of degree ϵ on Z defined by the algebra of endomorphisms of k^ϵ . Using the standard symplectic or orthogonal bilinear form on k^ϵ , one can construct a canonical symplectic or orthogonal involution on $\text{End}_k(k^\epsilon)$. We denote this involution by r . (We refer the reader to Appendix B for the details.) It is an easy exercise to show that r descends to $Z^s/G(\epsilon)^{ad}$. Hence we get an Azumaya algebra with symplectic or orthogonal involution (\mathcal{B}, r) on $Z^s/G(\epsilon)^{ad}$.

Finally, we need to construct g sections x_1, \dots, x_g of \mathcal{B} such that, for any field F and any map $\text{Spec } F \longrightarrow Z^s/G(\epsilon)^{ad}$, the pullback of \mathcal{B} along with r and the x_i give a stable CSA with involution over F . Recall that $Z = \mathfrak{g}(\epsilon)^{\times g}$. We define a section x_i of the split Azumaya algebra of degree ϵ on Z by first taking the i^{th} projection $Z \longrightarrow \mathfrak{g}(\epsilon)$ and composing with the inclusion $\mathfrak{g}(\epsilon) \longrightarrow \text{End}_k(k^\epsilon)$. Again, one can check that these sections descend to $Z^s/G(\epsilon)^{ad}$ and they define a stable CSA with involution.

5. THE FIELD-VALUED POINTS OF $Z^s/G(\epsilon)^{ad}$

In this section, for a field F containing k , we describe the F -valued points of $Z^s/G(\epsilon)^{ad}$ in terms of the stable central simple algebras (CSA) with involution defined in Section 4. Let Fields/k denote the category of field extensions of k . Let

$$\Phi_1 : \text{Fields}/k \longrightarrow (\text{Set})$$

be the functor that sends any F to the set of isomorphism classes of stable CSAs with involution over F . Given a morphism $\text{Spec } K \longrightarrow \text{Spec } F$, the corresponding map $\Phi_1(\text{Spec } F) \longrightarrow \Phi_1(\text{Spec } K)$ is defined by pull-back. We also consider the functor of points

$$\Phi_2 : \text{Fields}/k \longrightarrow (\text{Set})$$

that takes $\text{Spec } F$ to the set $\text{Mor}(\text{Spec } F, Z^s/G(\epsilon)^{ad})$.

Theorem 5.1. *The two functors Φ_1 and Φ_2 are naturally equivalent.*

Proof. There is a natural transformation $\alpha : \Phi_2 \longrightarrow \Phi_1$ that takes any $\text{Spec } F \longrightarrow Z^s/G(\epsilon)^{ad}$ to the pullback of \mathcal{B} to $\text{Spec } F$ via this morphism. We shall define another natural transformation $\beta : \Phi_1 \longrightarrow \Phi_2$ and prove that α and β are inverses to each other.

Let (A, σ, x_i) be a stable CSA with involution over the field F . Choose a finite Galois extension K/F splitting (A, σ, x_i) . Choose an isomorphism $A_K \xrightarrow{\sim} \text{End}(K^{2m}, Q_0)$. This gives a $\Lambda \otimes K$ -module structure on K^{2m} ; and from the definition of a stable CSA with involution, we have that this module has no nontrivial isotropic submodules. Hence by [6, Proposition 4.2], we obtain a map $\phi : \text{Spec } K \longrightarrow Z^s$. Consider the composition $\pi \circ \phi : \text{Spec } K \longrightarrow Z^s \longrightarrow Z^s/G(\epsilon)^{ad}$. We want to show that for every $\tau \in \text{Gal}(K/F)$, the diagram

$$(5.1) \quad \begin{array}{ccc} \text{Spec } K & \xrightarrow{\pi \circ \phi} & Z^s/G(\epsilon)^{ad} \\ \downarrow \tau & & \parallel \\ \text{Spec } K & \xrightarrow{\pi \circ \phi} & Z^s/G(\epsilon)^{ad} \end{array}$$

commutes and hence the map $\pi \circ \phi : \text{Spec } K \longrightarrow Z^s/G(\epsilon)^{ad}$ descends to give a map $\psi : \text{Spec } F \longrightarrow Z^s/G(\epsilon)^{ad}$.

Let $\{A_\tau\}$ be a 1-cocycle representing the class

$$[(A, \sigma)] \in H^1(\text{Gal}(K/F), G(\epsilon)^{ad}(K)).$$

Recall that this defines an action of $\text{Gal}(K/F)$ on A_K , and $A \subseteq A_K$ consists of the invariant elements. Since the elements x_i are in A , they are invariant under the $\text{Gal}(K/F)$ -action. Hence, we have

$$x_i = A_\tau(x_i)^\tau (A_\tau)^{-1}$$

for all $\tau \in \text{Gal}(K/F)$. Translating this into a commutative diagram, we get

$$(5.2) \quad \begin{array}{ccc} \text{Spec } K & \xrightarrow{\phi} & Z^s \\ \tau \downarrow & & \downarrow A_\tau(-)A_\tau^{-1} \\ \text{Spec } K & \xrightarrow{\phi} & Z^s, \end{array}$$

and composing with $\pi : Z^s \longrightarrow Z^s/G(\epsilon)^{ad}$ gives us exactly the diagram in (5.1). Hence we obtain the desired map $\psi : \text{Spec } F \longrightarrow Z^s/G(\epsilon)^{ad}$.

We are now going to prove that ψ is independent of the choice of the finite Galois extension K/F splitting (A, σ, x_i) as well as the choice of the isomorphism $(A, \sigma)_K \xrightarrow{\sim} \text{End}(K^{2m}, Q_0)$. Let

K_1/F and K_2/F be two such extensions, and let K/F be a finite Galois extension containing K_1 and K_2 . Choosing two isomorphisms $A_{K_i} \xrightarrow{\sim} \text{End}(K_i^{2m}, Q_0)$, $i = 1, 2$, and extending them to K , we obtain two maps $\phi_i : \text{Spec } K_i \rightarrow Z^s$ whose compositions with $j_i : \text{Spec } K \rightarrow \text{Spec } K_i$ differ by an element of $G(\epsilon)^{ad}(K)$. Hence we have that $\pi \circ \phi_1 \circ j_1 = \pi \circ \phi_2 \circ j_2$.

Now $\pi \circ \phi_1$ descends to $\psi_1 : \text{Spec } F \rightarrow Z^s/G(\epsilon)^{ad}$ as proved above. Hence $\pi \circ \phi_1 \circ j_1$ also descends to ψ_1 . Similarly, $\pi \circ \phi_2$ descends to $\psi_2 : \text{Spec } F \rightarrow Z^s/G(\epsilon)^{ad}$, and hence $\pi \circ \phi_2 \circ j_2$ descends to ψ_2 . Since $\pi \circ \phi_1 \circ j_1 = \pi \circ \phi_2 \circ j_2$, it follows that $\psi_1 = \psi_2$. This finishes the construction of β .

It remains to prove that α and β are inverses to each other. To prove that $\alpha \circ \beta = \text{Id}$, consider a stable CSA with involution (A, σ, x_i) over a field F , and choose a finite Galois extension K/F splitting it. We obtain a morphism $\phi : \text{Spec } K \rightarrow Z^s$ whose composition with π descends to $\psi : \text{Spec } F \rightarrow Z^s/G(\epsilon)^{ad}$. We note that $\pi \circ \phi$ pulls back \mathcal{B} to $(A, \sigma, x_i)_K$.

Consider the class $[(A, \sigma)] \in H^1(\text{Gal}(K/F), G(\epsilon)^{ad}(K))$. Let $\{A_\tau\}$ be a 1-cocycle representing this class. Since the x_i come from elements in A , we have $x_i = A_\tau(x_i)^\tau A_\tau^{-1}$; in other words, we have the commutative diagram in (5.2). Hence the action of $\text{Gal}(K/F)$ on $(A, \sigma, x_i)_K$ is the same as the action defined by the 1-cocycle $\{A_\tau\}$. This implies that \mathcal{B} pulls back to (A, σ, x_i) under $\psi : \text{Spec } F \rightarrow Z^s/G(\epsilon)^{ad}$, and hence that $\alpha \circ \beta = \text{Id}$.

To prove that $\beta \circ \alpha = \text{Id}$, take $\psi : \text{Spec } F \rightarrow Z^s/G(\epsilon)^{ad}$. Consider the stable CSA with involution obtained by pulling \mathcal{B} to $\text{Spec } F$ via ψ . Take a finite Galois extension K/F splitting $\psi^*\mathcal{B}$, and obtain a morphism $\phi : \text{Spec } K \rightarrow Z^s$. We then need to prove that the diagram

$$\begin{array}{ccc} \text{Spec } K & \xrightarrow{\phi} & Z^s \\ \downarrow & & \downarrow \pi \\ \text{Spec } F & \xrightarrow{\psi} & Z^s/G(\epsilon)^{ad} \end{array}$$

commutes. (Note that there is a slight ambiguity in the choice of ϕ here, due to the fact that one must choose an isomorphism $(\psi^*\mathcal{B})_K \rightarrow \text{End}(K^{2m}, Q_0)$. However, since such choices only affect ϕ up to conjugation by an element of $G(\epsilon)^{ad}$, this will not be a problem.)

This follows from the following claim: Any morphism $f : \text{Spec } M \rightarrow Z^s/G(\epsilon)^{ad}$ that pulls \mathcal{B} back to a *split* stable CSA with involution, lifts to a morphism $g : \text{Spec } M \rightarrow Z^s$.

Proof of the claim: Consider a morphism $f : \text{Spec } M \rightarrow Z^s/G(\epsilon)^{ad}$ that pulls \mathcal{B} back to a split stable CSA with involution, giving a morphism $g : \text{Spec } M \rightarrow Z^s$ that pulls \mathcal{A} back to the stable CSA with involution on $\text{Spec } M$. Let \overline{M} denote the algebraic closure of M . The composition of f with $\text{Spec } \overline{M} \rightarrow \text{Spec } M$ lifts to Z^s , i.e., the outer square in the diagram

$$\begin{array}{ccc} \text{Spec } \overline{M} & \xrightarrow{\overline{f}} & Z^s \\ \downarrow i & \nearrow g & \downarrow \pi \\ \text{Spec } M & \xrightarrow{f} & Z^s/G(\epsilon)^{ad} \end{array}$$

commutes. Now it is easily checked that $g \circ i$ and \overline{f} pull the stable CSA with involution \mathcal{A} back to isomorphic stable CSAs with involution over \overline{M} . Hence $g \circ i = \overline{f}$; and this implies that $f = \pi \circ g$, finishing the proof of the claim and the proof of the theorem. \square

6. CONSTRUCTION OF STABLE CSAs WITH INVOLUTION

Throughout this section, we will write $\epsilon = 2^\alpha s$. We also remind the reader that $g \geq 2$, where g was defined in Remark 2.10.

Given a field L and two elements $\alpha, \beta \in L \setminus L^{\times 2}$ we denote by (α, β) the quaternion algebra formed by taking square roots of α and β . Concretely, this is the subalgebra of

$$M_2(L(\sqrt{\alpha}, \sqrt{\beta}))$$

generated by the matrices

$$\begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & -\sqrt{\alpha} \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}.$$

The reader is referred to Appendix B for more details.

Let $F = k(x_1, \dots, x_\alpha, y_1, \dots, y_\alpha)$ and $K = k(\sqrt{x_1}, \dots, \sqrt{x_\alpha}, y_1, \dots, y_\alpha)$.

We let

$$D = (x_1, y_1) \otimes \dots \otimes (x_\alpha, y_\alpha)$$

Theorem 6.1. *The central simple algebra D over F is a division algebra and hence has index 2^α .*

Proof. See [1, Theorem 3]. □

6.1. The symplectic case with α odd. Recall from the appendix B that the quaternion algebras (x_i, y_i) have a natural symplectic involution that we denoted by σ_i .

Suppose that α is odd and recall that $\epsilon = 2^\alpha s$. The central simple algebra $D \otimes M_s(F)$ has a symplectic involution

$$\sigma_1 \otimes \sigma_2 \otimes \dots \otimes \sigma_\alpha \otimes t$$

where t is the transpose involution on the matrix algebra. (See [10, Proposition 2.23].)

Proposition 6.2. *Suppose that α is odd. Denote by i_n (respectively, j_n) the square roots of x_n (respectively, y_n) in the algebra*

$$D \otimes M_s(F).$$

If

$$\begin{aligned} A &= i_1 \otimes i_2 \otimes \dots \otimes i_\alpha \otimes \text{diag}(1, 2, \dots, s) \\ B &= j_1 \otimes j_2 \otimes \dots \otimes j_\alpha \otimes I_s \end{aligned}$$

then the collection of elements $\lambda_1 = A, \lambda_2 = x_3 = \dots = \lambda_g = B$ gives D the structure of a stable central simple algebra with symplectic involution.

Proof. In what follows, we will think of D as a subalgebra of a matrix algebra over K in the usual way.

It is easily checked that $\sigma(A) = -A$ and $\sigma(B) = -B$.

Now consider standard bases $\{e_1, e_2\}, \dots, \{e_{2\alpha-1}, e_{2\alpha}\}, \{f_1, \dots, f_s\}$ of the vector spaces K^2 and K^s . With respect to these standard bases, the eigenvalues of A are

$$\pm\sqrt{x_1 \cdots x_\alpha}, \dots, \pm s\sqrt{x_1 \cdots x_\alpha}.$$

Consider the eigenvalue $\sqrt{x_1 \cdots x_\alpha}$. (The proof for the other eigenvalues is similar and shall be omitted.) The eigenspace is spanned by the vectors $e_{i_1} \otimes \dots \otimes e_{i_\alpha} \otimes f_1$, where i_k is either $2k-1$ or $2k$ and an even number of the i_k 's are even. For simplicity, we denote $e_{i_1} \otimes \dots \otimes e_{i_\alpha} \otimes f_1$ by e_{i_1, \dots, i_α} .

Standard arguments show that if $v \in M$ has a non-zero projection onto an eigenspace then M must contain a non-zero eigenvector for that eigenspace.

Consider a nonzero $\Lambda \otimes K$ -submodule M of K^ϵ . Take a nonzero vector

$$v = \sum \lambda_{i_1, \dots, i_\alpha} e_{i_1, \dots, i_\alpha} \in M, \quad \lambda_{i_1, \dots, i_\alpha} \in K$$

that is an eigenvector for $\sqrt{x_1 \cdots x_\alpha}$. Then we have

$$Bv = \sum \lambda_{i_1, \dots, i_\alpha} y_1^{b_1} \cdots y_\alpha^{b_\alpha} e_{\overline{i_1}, \dots, \overline{i_\alpha}},$$

where $\overline{i_k} = 2k - 1$ if $i_k = 2k$ and $\overline{i_k} = 2k$ if $i_k = 2k - 1$. Also, $b_k = 0$ if i_k is odd and $b_k = 1$ if i_k is even. Hence an even number of the b_k 's are 1.

The symplectic involution σ is adjoint with respect to the symplectic form given by $Q = \Sigma \otimes \cdots \Sigma \otimes I_s$. We claim that $Q(v, Bv) \neq 0$, which proves that the $\Lambda \otimes K$ -submodule M of K^ϵ is not isotropic, and that we have a stable CSA with involution $(D \otimes M_s(F), \sigma, \{x_1, \dots, x_g\})$.

For each term $\lambda_{i_1, \dots, i_\alpha} e_{i_1, \dots, i_\alpha}$ in v , the only term in Bv for which $Q(-, -)$ is nonzero is $\lambda_{i_1, \dots, i_\alpha} y_1^{b_1} \cdots y_\alpha^{b_\alpha} e_{\overline{i_1}, \dots, \overline{i_\alpha}}$. Hence, we have

$$Q(v, Bv) = \sum \lambda_{i_1, \dots, i_\alpha}^2 y_1^{b_1} \cdots y_\alpha^{b_\alpha}.$$

Assume that $Q(v, Bv) = 0$. Then we have the equation

$$\sum \lambda_{i_1, \dots, i_\alpha}^2 y_1^{b_1} \cdots y_\alpha^{b_\alpha} = 0.$$

Using Lemma A.1 we see that this is a contradiction, hence finishing the proof. \square

6.2. The symplectic case with α even. The quaternion algebra (x_α, y_α) has an orthogonal involution τ , described in the appendix. The involution

$$\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_{\alpha-1} \otimes \tau \otimes t$$

is a symplectic involution on $D \otimes M_s(F)$. (See [10, Proposition 2.23].)

Proposition 6.3. *Suppose that α is even. Denote by i_n (respectively, j_n) the square roots of x_n (respectively, y_n) in the algebra*

$$D \otimes M_s(F).$$

If

$$\begin{aligned} A &= i_1 \otimes i_2 \otimes \cdots \otimes i_\alpha \otimes \text{diag}(1, 2, \dots, s) \\ B &= j_1 \otimes j_2 \otimes \cdots \otimes j_{\alpha-1} \otimes 1 \otimes I_s \end{aligned}$$

then the collection of elements $\lambda_1 = A, \lambda_2 = \lambda_3 = \dots = \lambda_g = B$ gives D the structure of a stable central simple algebra with symplectic involution.

Proof. It is easily checked that $\sigma(A) = -A$ and $\sigma(B) = -B$.

The eigenvalues of A are as in the proof of Proposition 6.2, and we keep the notation. Consider a nonzero $\Lambda \otimes K$ -submodule M of K^ϵ . Take a nonzero vector

$$v = \sum \lambda_{i_1, \dots, i_\alpha} e_{i_1, \dots, i_\alpha} \in M, \quad \lambda_{i_1, \dots, i_\alpha} \in K,$$

that is an eigenvector for $\sqrt{x_1 \cdots x_\alpha}$. (The cases of the other eigenvalues are similar and are left to the reader.) Then we have

$$Bv = \sum \lambda_{i_1, \dots, i_\alpha} y_1^{b_1} \cdots y_{\alpha-1}^{b_{\alpha-1}} e_{\overline{i_1}, \dots, \overline{i_{\alpha-1}}, i_\alpha}.$$

The symplectic involution σ is adjoint with respect to the symplectic form given by $Q = \Sigma \otimes \cdots \Sigma \otimes T \otimes I_s$. We claim that $Q(v, Bv) \neq 0$, which proves that the $\Lambda \otimes K$ -submodule M of K^{2m} is not isotropic, and that we have a stable CSA with involution $(D \otimes M_s(F), \sigma, \{x_1, \dots, x_g\})$.

¹We keep this notation throughout the rest of the section.

For each term $\lambda_{i_1, \dots, i_\alpha} e_{i_1, \dots, i_\alpha}$ in v , the only term in Bv for which $Q(-, -)$ is nonzero is $\lambda_{i_1, \dots, i_\alpha} y_1^{b_1} \cdots y_{\alpha-1}^{b_{\alpha-1}} e_{\overline{i_1}, \dots, \overline{i_{\alpha-1}}, i_\alpha}$. Hence, we have

$$\begin{aligned}
Q(v, Bv) = & \sum_{\substack{i_\alpha \text{ is odd,} \\ \text{an even number of } i_1, \dots, i_{\alpha-1} \text{ are even}}} \lambda_{i_1, \dots, i_\alpha}^2 y_1^{b_1} \cdots y_{\alpha-1}^{b_{\alpha-1}} \frac{1}{\sqrt{x_\alpha}} \\
& \pm \sum_{\substack{i_\alpha \text{ is even,} \\ \text{an odd number of } i_1, \dots, i_{\alpha-1} \text{ are even}}} \lambda_{i_1, \dots, i_\alpha}^2 y_1^{b_1} \cdots y_{\alpha-1}^{b_{\alpha-1}} \frac{1}{y_\alpha \sqrt{x_\alpha}}.
\end{aligned}$$

Above, the \pm sign is determined by the parity of the i_k 's. Assume $Q(v, Bv) = 0$. Then we have

$$\begin{aligned}
& \sum_{\substack{i_\alpha \text{ is odd,} \\ \text{an even number of } i_1, \dots, i_{\alpha-1} \text{ are even}}} \lambda_{i_1, \dots, i_\alpha}^2 y_1^{b_1} \cdots y_{\alpha-1}^{b_{\alpha-1}} \\
= & \sum_{\substack{i_\alpha \text{ is even,} \\ \text{an odd number of } i_1, \dots, i_{\alpha-1} \text{ are even}}} \lambda_{i_1, \dots, i_\alpha}^2 y_1^{b_1} \cdots y_{\alpha-1}^{b_{\alpha-1}} \frac{1}{y_\alpha}.
\end{aligned}$$

(Above, we incorporate the possible $-$ sign into the $\lambda_{i_1, \dots, i_\alpha}^2$ since the base field k contains a square root of -1 .) Multiplying both sides by y_α , and using Lemma A.1 as before we see that this is a contradiction, hence finishing the proof. \square

6.3. The orthogonal case with α odd and $s \neq 1$. Recall the definition of the involution δ on a quaternion algebra from the appendix. The involution

$$\sigma = \delta_1 \otimes \cdots \otimes \delta_\alpha \otimes t$$

on $D \otimes M_s(F)$ is orthogonal by [10, Proposition 2.23].

Consider the following elements of $D \otimes M_s(F)$:

$$\begin{aligned}
A &= i_1 \otimes \cdots \otimes i_\alpha \otimes \text{diag}(1, \dots, s) \\
B &= (i_1 \otimes \cdots \otimes i_\alpha \otimes M_1) + (j_1 \otimes \cdots \otimes j_\alpha \otimes M_2)
\end{aligned}$$

of $D \otimes M_s(F)$, where we have $M_1, M_2 \in M_s(F)$ defined as

$$M_1 = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix},$$

and

$$M_2 = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 \end{pmatrix}.$$

Proposition 6.4. *Suppose α odd and $s \neq 1$. In the above notation the system of elements $\lambda_1 = A$ and $\lambda_2 = \lambda_3 = \cdots \lambda_g = B$ gives $D \otimes M_s(F)$ the structure of a stable central simple algebra with orthogonal involution.*

Proof. It is easily checked that $\sigma(A) = -A$ and $\sigma(B) = -B$.

The eigenvalues of A are as in the proof of Proposition 6.2, and we keep the notation. Consider a nonzero $\Lambda \otimes K$ -submodule M of K^{2m} . Take a nonzero vector $v = \sum \lambda_{i_1, \dots, i_\alpha} e_{i_1, \dots, i_\alpha} \in M$, $\lambda_{i_1, \dots, i_\alpha} \in K$, that is an eigenvector for $\sqrt{x_1 \cdots x_\alpha}$. (The cases of the other eigenvalues are similar and are left to the reader.) Then we have

$$\begin{aligned} Bv &= \sum \lambda_{i_1, \dots, i_\alpha} (\sqrt{x_1 \cdots x_\alpha} e_{i_1} \otimes \cdots \otimes e_{i_\alpha} \otimes (f_1 + \cdots + f_s) \\ &\quad + y_1^{b_1} \cdots y_\alpha^{b_\alpha} e_{i_1}^{-1} \otimes \cdots \otimes e_{i_\alpha}^{-1} \otimes (-f_2 - \cdots - f_s)). \end{aligned}$$

The orthogonal involution σ is adjoint with respect to the symplectic form given by $Q = \Delta \otimes \cdots \otimes \Delta \otimes I_s$. We claim that $Q(Bv, Bv) \neq 0$, which proves that the $\Lambda \otimes K$ -submodule M of K^{2m} is not isotropic, and that we have a stable CSA with involution $(D \otimes M_s(F), \sigma, \{x_1, \dots, x_g\})$.

Indeed, one computes

$$Q(Bv, Bv) = 2(-s+1) \sum \lambda_{i_1, \dots, i_\alpha}^2 y_1^{b_1} \cdots y_\alpha^{b_\alpha} \sqrt{x_1 \cdots x_\alpha}.$$

By assumption, $-s+1 \neq 0$. Hence if one assumes that $Q(Bv, Bv) = 0$, one obtains a contradiction using Lemma A.1. This finishes the proof. \square

6.4. The orthogonal case with α even and $s \neq 1$. The involution

$$\sigma = \delta_1 \otimes \cdots \otimes \delta_{\alpha-1} \otimes t_\alpha \otimes t$$

on $D \otimes M_s(F)$ is orthogonal by [10, Proposition 2.23].

Define two elements

$$\begin{aligned} A &= i_1 \otimes \cdots \otimes i_\alpha \otimes \text{diag}(1, \dots, s) \\ B &= (i_1 \otimes \cdots \otimes i_\alpha \otimes M_1) + (j_1 \otimes \cdots \otimes j_{\alpha-1} \otimes 1 \otimes M_2) \end{aligned}$$

of $D \otimes M_s(F)$, where M_1 and M_2 are as in the previous subsection.

Proposition 6.5. *Suppose α even and $s \neq 1$. In the above notation the system of elements $\lambda_1 = A$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_g = B$ gives $D \otimes M_s(F)$ the structure of a stable central simple algebra with orthogonal involution.*

Proof. It is easily checked that $\sigma(A) = -A$ and $\sigma(B) = -B$.

The eigenvalues of A are as in the proof of Proposition 6.2, and we keep the notation. Consider a nonzero $\Lambda \otimes K$ -submodule M of K^{2m} . Take a nonzero vector

$$v = \sum \lambda_{i_1, \dots, i_\alpha} e_{i_1, \dots, i_\alpha} \in M, \quad \lambda_{i_1, \dots, i_\alpha} \in K,$$

that is an eigenvector for $\sqrt{x_1 \cdots x_\alpha}$. (The cases of the other eigenvalues are similar and are left to the reader.) Then we have

$$\begin{aligned} Bv &= \sum \lambda_{i_1, \dots, i_\alpha} (\sqrt{x_1 \cdots x_\alpha} e_{i_1} \otimes \cdots \otimes e_{i_\alpha} \otimes (f_1 + \cdots + f_s) \\ &\quad + y_1^{b_1} \cdots y_\alpha^{b_\alpha} e_{i_1}^{-1} \otimes \cdots \otimes e_{i_{\alpha-1}}^{-1} \otimes e_{i_\alpha} \otimes (-f_2 - \cdots - f_s)). \end{aligned}$$

The orthogonal involution σ is adjoint with respect to the symplectic form given by $Q = \Delta^{\otimes \alpha-1} \otimes I_2 \otimes I_s$. We claim that $Q(Bv, Bv) \neq 0$, which proves that the $\Lambda \otimes K$ -submodule M of K^{2m} is not isotropic, and that we have a stable CSA with involution $(D \otimes M_m(F), \sigma, \{x_1, \dots, x_g\})$.

Indeed, one computes

$$Q(Bv, Bv) = 2(-s+1) \sum \lambda_{i_1, \dots, i_\alpha}^2 y_1^{b_1} \cdots y_\alpha^{b_\alpha} \sqrt{x_1 \cdots x_\alpha}.$$

By assumption, $-s+1 \neq 0$. Hence if one assumes $Q(Bv, Bv) = 0$, one obtains a contradiction using Lemma A.1. This finishes the proof. \square

Remark 6.6. In the proofs of Propositions 6.4 and 6.5, if one replaces the factor (x_α, y_α) in D by $M_2(F)$ and δ_α t_α , while F and K are changed so that they have $\alpha - 1$ number of x and y variables; the same proofs carry through and hence there exists a stable CSA with involution $((x_1, y_1) \otimes \cdots \otimes (x_{\alpha-1}, y_{\alpha-1}) \otimes M_2(F), \sigma, \{x_1, \dots, x_g\})$ over F .

7. THE MAIN RESULTS

Recall that $\epsilon = 2^\alpha s$ with s odd. We also remind the reader that $g \geq 2$, where g was defined in Remark 2.10. Recall that $F = k(x_1, \dots, x_\alpha, y_1, \dots, y_\alpha)$ and $K = k(\sqrt{x_1}, \dots, \sqrt{x_\alpha}, y_1, \dots, y_\alpha)$.

Theorem 7.1.

- (1) For $G(\epsilon) = \mathrm{Sp}(\epsilon)$ or $G(\epsilon) = \mathrm{SO}(\epsilon)$ with $s > 1$, there exists a stable CSA with involution $(D \otimes M_s(F), \sigma, \{x_1, \dots, x_g\})$ over F .
- (2) If $G(\epsilon) = \mathrm{SO}(\epsilon)$ with $\epsilon = 2^\alpha$, then there exists a stable CSA with involution $((x_1, y_1) \otimes \cdots \otimes (x_{\alpha-1}, y_{\alpha-1}) \otimes M_2(F), \sigma, \{x_1, \dots, x_g\})$ over F .

Proof. See Propositions 6.2, 6.3, 6.4, 6.5 and Remark 6.6. \square

Corollary 7.2. Recall the definition of the CSA \mathcal{B} from Example 4.6.

- (1) For $G(\epsilon) = \mathrm{Sp}(\epsilon)$ or $G(\epsilon) = \mathrm{SO}(\epsilon)$ with $s > 1$, the index of \mathcal{B} is divisible by 2^α .
- (2) For $G(\epsilon) = \mathrm{SO}(\epsilon)$ with $\epsilon = 2^\alpha$, the index of \mathcal{B} is divisible by $2^{\alpha-1}$.

Proof. Combine Theorem 5.1 with Theorem 6.1 and Theorem 7.1. \square

Let $\widehat{F} = k((x_1, y_1, \dots, x_\alpha, y_\alpha))$ and $\widehat{K} = k((\sqrt{x_1}, y_1, \dots, \sqrt{x_\alpha}, y_\alpha))$. (In the case of $G(\epsilon) = \mathrm{SO}(\epsilon)$ with $\epsilon = 2^\alpha$, it is understood that there would be $\alpha - 1$ number of x and y variables in the definition. We will assume this tacitly to avoid repetition in the below proof.) Let ψ_0 denote the maps $\mathrm{Spec} F \rightarrow Z^s/G(\epsilon)^{ad}$ constructed in the proof of Theorem 7.1. Composing ψ_0 with the canonical map $\mathrm{Spec}(\widehat{F}) \rightarrow \mathrm{Spec}(F)$, we obtain a map

$$\phi_0 : \mathrm{Spec}(\widehat{F}) \rightarrow Z^s/G(\epsilon)^{ad}.$$

We have

$$\phi_0^*(\mathcal{A}) \cong (x_1, y_1) \otimes \cdots \otimes (x_\alpha, y_\alpha) \otimes M_s(\widehat{F}).$$

Recall the open subscheme

$$\mathrm{Spec}(\widehat{\mathcal{O}}_0)^t$$

constructed in Section 2.

Proposition 7.3. We have a factorization :

$$\begin{array}{ccc} \mathrm{Spec}(\widehat{F}) & & \\ \downarrow & \searrow & \\ \mathrm{Spec}(\widehat{\mathcal{O}}_0)^t & \longrightarrow & Z^s/G(\epsilon)^{ad}. \end{array}$$

Proof. Firstly the map ψ_0 factors through $(Z//G)^t$ as it factors through the stable locus as $(Z//G)^t$ is dense in $(Z//G)^s$, being an open subset of an irreducible set. So it suffices to show that ϕ_0 factor through the completion of the local ring at 0.

The remainder of the proof is essentially the same as the proof of Corollary 6.2 in [6].

We take $R = k[\sqrt{x_1}, \dots, \sqrt{x_\alpha}, y_1, \dots, y_\alpha]$. As the formulas for the λ_i in Propositions 6.2, 6.3, 6.4 and 6.5 do not involve denominators there is a diagram

$$\begin{array}{ccc} \text{Spec}(R) & & \\ \uparrow & \searrow \Phi & \\ \text{Spec}(K) & & Z \\ \downarrow & & \downarrow \\ \text{Spec}(F) & \longrightarrow & Z//G(\epsilon) \end{array}$$

Denote $\Phi(0) = P \in Z$. Hence we get a map on completions of local rings, and we obtain a map

$$\phi_c : \text{Spec}(\widehat{K}) \longrightarrow \widehat{\mathcal{O}_{Z,P}}.$$

In view of Theorem 7.1, it is enough to show that it descends to a morphism $\text{Spec}(\widehat{F}) \longrightarrow \widehat{\mathcal{O}_{Z//G,0}^s}$. We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{Z//G(\epsilon),0} & \longrightarrow & \widehat{F} \\ \downarrow & & \downarrow \\ \mathcal{O}_{Z,P} & \longrightarrow & \widehat{K} \end{array}$$

with ϕ_c induced by completion from the bottom line. The image of the maximal ideal of $\mathcal{O}_{Z//G(\epsilon),0}$ must be contained in the $\mathcal{O}_{Z//G(\epsilon),0}$ submodule of \widehat{F} generated by $(x_1, y_1, \dots, x_\alpha, y_\alpha)$. But the field \widehat{F} is complete with respect to the induced topology and hence we obtain our map. \square

Theorem 7.4 (Index of the canonical gerbe).

- (1) For $G(\epsilon) = \text{Sp}(2n)$ or $G(\epsilon) = \text{SO}(2n)$ with $s > 1$, the index of the canonical gerbe $\text{Bun}_G^{\text{rs}} \longrightarrow M_G^{\text{rs}}$ is 2^α .
- (2) For $G(\epsilon) = \text{SO}(2n)$ with $\epsilon = 2^\alpha$, the index of the canonical gerbe $\text{Bun}_G^{\text{rs}} \longrightarrow M_G^{\text{rs}}$ is $2^{\alpha-1}$ or 2^α .

Proof. By Propositions 2.11 and 7.3, the index of the canonical gerbe is divisible by 2^α (or $2^{\alpha-1}$ in the second case). By Proposition 3.4, the index of the canonical gerbe divides ϵ and hence it divides 2^α . The result now follows from Corollary 7.2. \square

APPENDIX A. AN ANISOTROPIC FORM OVER THE FIELD OF RATIONAL FUNCTIONS

The following well-known result was used in the construction of stable CSAs with involution.

Lemma A.1. *The n -Pfister form $\ll t_1, \dots, t_n \gg$ over the field $k(t_1, \dots, t_n)$ is anisotropic.*

Proof. We use induction on n . For $n = 1$, the form $\ll t_1 \gg = \langle 1, t_1 \rangle$ has no isotropic vectors as the equation

$$f_1^2 + t_1 f_2^2 = 0$$

implies that t_1 is a square in $k(t_1, \dots, t_n)$, a contradiction.

Assume that the statement is proved for $(n - 1)$ -Pfister forms, and consider the n -Pfister form $\ll t_1, \dots, t_n \gg$. Assume that there exists an isotropic vector for $\ll t_1, \dots, t_n \gg$, hence an equation

$$\sum_{I \subset \{1, \dots, n\}} t_I f_I^2 = 0,$$

where I runs over all subsets of $\{1, \dots, n\}$, t_I is the monomial obtained by multiplying the t_i for which $i \in I$ and, by clearing denominators and removing common factors, we assume that the f_I are polynomials with no common factors.

By setting $t_n = 0$, we have $\sum_{I \subset \{1, \dots, n-1\}} t_I \overline{f_I}^2 = 0$, where $\overline{f_I}$ denotes the reduction of f_I modulo t_n . By the induction hypothesis, this can only happen if all the $\overline{f_I}$ are zero; i.e., when t_n divides the f_I for $I \subset \{1, \dots, n-1\}$.

Now rearrange the equation above to get

$$t_n^2 g = -t_n \sum_{I \subset \{1, \dots, n-1\}} t_{I \cup \{n\}} f_{I \cup \{n\}}^2$$

for some polynomial g . After cancelling t_n and setting $t_n = 0$ again, we obtain a similar equation

$$\sum_{I \subset \{1, \dots, n-1\}} t_{I \cup \{n\}} \overline{f_{I \cup \{n\}}}^2 = 0,$$

which again implies that t_n divides the remaining f_I . Hence t_n divides all the f_I , which is a contradiction. \square

APPENDIX B. CENTRAL SIMPLE ALGEBRAS WITH INVOLUTION

Let k be a field. Recall that a *central simple algebra* over k is a k -algebra A which is finite-dimensional as a k -vector space, whose center is k (viewed as a subring of A) and which has no proper, non-trivial two-sided ideals. Given a central simple algebra A over k , there exists a finite field extension L of k such that $A_L = A \otimes_k L$ is isomorphic to a matrix algebra $M_n(L)$ over L . Hence, the dimension of A over k is a square n^2 . The number n is called the *degree* of A .

Given two central simple algebras A and B over k , we call A and B *Brauer-equivalent* if there exist natural numbers m and n such that $M_m(k) \otimes_k A \cong M_n(k) \otimes_k B$. The set of Brauer-equivalence classes of central simple algebras has the structure of an abelian group, denoted $\text{Br}(k)$, described as follows. Multiplication of two elements $[A], [B] \in \text{Br}(k)$ is given by $[A][B] = [A \otimes_k B]$, the identity element is given by the equivalence class $[k]$ of k itself, and inverses are given by $[A]^{-1} = [A^\circ]$, where A° is the opposite algebra of A . $\text{Br}(k)$ is a torsion abelian group, and the order of an element of $a \in \text{Br}(k)$ is called the *period* of a .

By a theorem of Wedderburn, every central simple algebra A over k can be written as $M_n(D)$, where D is a *central division algebra* over k , meaning a central simple algebra over k that is a division ring. D is unique up to isomorphism. Hence, the degree of D is well-defined, and is called the *index* of A . Two Brauer-equivalent central simple algebras A and B over k have the same index, hence the index is defined for elements of $\text{Br}(k)$.

There is a natural isomorphism

$$\text{Br}(K) \xrightarrow{\text{Sym}} \text{H}^2(K, \mathbb{G}_m),$$

and hence associated to every central simple algebra is a gerbe. The notion of index and period defined here agrees with the one in Section 3. Hence by Proposition 3.2, we have that the period divides the index, and that the period and the index have the same prime powers.

An *involution of the first kind* on a central simple algebra A over k is an additive map $\sigma : A \rightarrow A$ such that $\sigma(xy) = \sigma(y)\sigma(x)$, $\sigma^2 = \text{Id}_A$ and $\sigma(\lambda) = \lambda$ for all $\lambda \in k$. From now on, we will refer to an involution of the first kind as simply an involution.

Consider the central simple algebra $M_n(k)$ over k , which can also be viewed as $\text{End}_k(V)$, where V is an n -dimensional vector space over k . Then there is a one-to-one correspondence between involutions on $\text{End}_k(V)$ and equivalence classes of nonsingular bilinear forms on V modulo multiplication by an element of k^\times that are either symmetric or skew-symmetric. (See the Theorem in the introduction to [10, Chapter 1].) Let b be a symmetric or skew-symmetric bilinear form on V , and σ the corresponding involution on $\text{End}_k(V)$. Fix an ordered basis for V and denote the Gram matrix of b with respect to this basis by $g \in \text{GL}_n(k)$. Here, $g^t = g$ if b is symmetric and $g^t = -g$ if b is skew-symmetric. Then the involution σ is given by

$$\sigma(m) = g^{-1}m^t g$$

for $m \in M_n(k)$.

Let A be a central simple algebra over k , and σ an involution on A . Choose a field extension L of k that splits A , i.e., $A_L = M_n(L)$. Over this base extension, consider the bilinear form b that corresponds to the involution $\sigma_L = \sigma \otimes_k \text{Id}_L$. If b is symmetric, σ is called *orthogonal*; and if b is skew-symmetric, σ is called *symplectic*.

Write $2n = 2^\alpha m$ where m is odd. Let

$$F = \mathbb{k}(x_1, y_1, \dots, x_\alpha, y_\alpha) \quad \text{and} \quad K = \mathbb{k}(\sqrt{x_1}, y_1, \dots, \sqrt{x_\alpha}, y_\alpha).$$

Then K/F is a Galois extension with Galois group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\alpha$.

For $\ell = 1, \dots, \alpha$, let (x_ℓ, y_ℓ) denote the quaternion algebra over F having a basis $\{1, i, j, k\}$ such that $i^2 = x_\ell$, $j^2 = y_\ell$ and $k = ij = -ji$. The *quaternion conjugation* or *canonical involution* is the F -linear map $\sigma : (x_\ell, y_\ell) \rightarrow (x_\ell, y_\ell)$ given by $a + bi + cj + dk \mapsto a - bi - cj - dk$. By [10, Proposition 2.21], the canonical involution is the only symplectic involution on (x_ℓ, y_ℓ) . Note that over K , (x_ℓ, y_ℓ) splits, and we have $\sigma_K = \text{Int}(\Sigma) \circ t$; where

$$\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and $t : M_2(K) \rightarrow M_2(K)$ denotes the transpose involution. For later reference, we note that over K , i_ℓ and j_ℓ are represented by the matrices

$$\begin{pmatrix} \sqrt{x_\ell} & 0 \\ 0 & -\sqrt{x_\ell} \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 \\ y_\ell & 0 \end{pmatrix}.$$

We will also need two orthogonal involutions on (x_ℓ, y_ℓ) . Define $\tau = \text{Int}(k) \circ \sigma$ and $\delta = \text{Int}(i) \circ \sigma$. Then τ is given by $a + bi + cj + dk \mapsto a + bi + cj - dk$, δ is given by $a + bi + cj + dk \mapsto a - bi + cj + dk$ and they are orthogonal involutions by [10, Proposition 2.21]. After splitting (x_ℓ, y_ℓ) by extending the base field to K , $\tau = \text{Int}(T) \circ t$ and $\delta = \text{Int}(\Delta) \circ t$, where

$$T = \begin{pmatrix} -\sqrt{x_\ell} & 0 \\ 0 & -y_\ell \sqrt{x_\ell} \end{pmatrix},$$

and

$$\Delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We note that on $M_m(K)$, the transpose involution t is orthogonal.

The involutions $t \otimes_F \text{Id}_K$, $\sigma \otimes_F \text{Id}_K$, $\tau \otimes_F \text{Id}_K$ and $\delta \otimes_F \text{Id}_K$ on $(x_\ell, y_\ell) \otimes_F K$ are the adjoint involutions with respect to the following forms: $t \otimes_F \text{Id}_K$ is adjoint with respect to the orthogonal form represented with the identity matrix I_m , $\sigma \otimes_F \text{Id}_K$ is adjoint with respect to the symplectic form represented by the matrix

$$\Sigma^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$\tau \otimes_F \text{Id}_K$ is adjoint with respect to the orthogonal form represented by the matrix

$$T^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{x_\ell}} & 0 \\ 0 & -\frac{1}{y_\ell \sqrt{x_\ell}} \end{pmatrix},$$

and $\delta \otimes_F \text{Id}_K$ is adjoint with respect to the orthogonal form represented by the matrix

$$\Delta^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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